Fast Linear Solver Using Deflation Method and Multigrid Method for Optimization

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Fast liner solvers for shape optimization using a deflation technique and a multigrid method are discussed. The optimization method based on evolutionary algorithms such as a genetic algorithm requires huge computational cost to evaluate many trial shapes. In this reason, a deflated preconditioned conjugate gradient (PCG) method is introduced so as to reduce the cost of finite element analysis which is used to evaluate the objective function. The deflation technique decomposes the solution into fast and slowly components. The slowly components can be solved by direct methods with low computational cost due to small dimensions. Therefore the deflated PCC method can improve the convergence of PCG. However, the deflated PCG requires eigenvectors which have high computational cost. In this study, a multigrid method is introduced to solve this difficulty. Instead of the high cost eigenvectors, eigenvectors of the system matrix on a coarse mesh are used for the deflation method. The computational cost to obtain them is low because the system matrix on the coarse mesh is small. Thus the proposed method can reduce the computational cost. Numerical results show that the present method can improve the convergence and reduce the computational cost of optimizations.

***Index Terms*— Conjugate gradient method, Deflation method, Multigrid method, Topology optimization**

I. Introduction

FAST LINEAR SOLVERS are an important factor to reduce computation cost of finite element analyses. In the shape optimization for electromagnetic devices, the finite element analysis occupies almost all computational cost in the optimization process. Therefore, it is important to reduce the cost in solver process by introducing fast linear solvers.

We have studied topology optimization methods based on ON/OFF method and evolutionary algorithms. Finite element meshes which consist of lattice grids are required for the method. This often brings flat elements in air regions. These flat elements produce large condition number defined by the ratio of the largest eigenvalue to the smallest non-zero eigenvalue of system matrix. This results in poor convergence of preconditioned conjugate gradient such as incomplete Cholesky conjugate gradient (ICCG) method.

The deflation technique which replaces small eigenvalues with zeros in the system matrix can improve the convergence of ICCG method [1],[2]. Therefore the deflated ICCG method is a useful solver for such ill-conditioned finite element analyses. However, the deflated ICCG requires eigenvectors which have high computational cost. To solve this difficulty, simple quasi vectors instead of eigenvectors can be used [1]. However, the improvement of convergence is limited, because few numbers of quasi vectors can be obtained. In this study, an idea of multigrid method is introduced. The eigenvectors of the system matrix on a coarse mesh is computed. These vectors are projected onto finer meshes by using a prolongation operator. The projected eigenvectors are used for the deflated method. The cost to compute these procedures is low because of small system matrix on the coarse mesh. Thus the proposed method can reduce the computational cost.

The present method is applied to a topology optimization of power inductor model [3]. The efficiency of the present method in non-linear as well as linear magnetostatic analyses is investigated.

II. Formulation

Let us consider a system of linear equations with n DoF (Degree of Freedom) obtained by a magnetostatic finite element analysis.

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

where A is coefficient matrix, $\boldsymbol{x}$ and $\boldsymbol{b}$ are solution and right hand side vector respectively. The solution $\boldsymbol{x}$ is decomposed into slowly and fast components as,

$$\boldsymbol{x} = W\boldsymbol{y} + (\boldsymbol{x} - W\boldsymbol{y}) \tag{2}$$

where $W = [\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_k] \in \Re^{n\times k}$, $\boldsymbol{w}_i\ (i = 1,2,..,k) \in \Re^n$. A-orthogonality is imposed on the vectors $\boldsymbol{w}_i$ to $\boldsymbol{x} - W\boldsymbol{y}$ results in

$$W^tAW\boldsymbol{y} = W^tA\boldsymbol{x}. \tag{3}$$

The slowly converging component $W\boldsymbol{y}$ can be expressed as

$$W\boldsymbol{y} = W(W^tAW)^{-1}W^tA\boldsymbol{x} \equiv Q\boldsymbol{x} \tag{4}$$

where $Q \in \Re^{n\times n}$. Moreover, let us introduce the matrix P given by,

$$P = I - Q = I - W(W^tAW)^{-1}W^tA. \tag{5}$$

Consequently, the solution $\boldsymbol{x}$ can be expressed as,

$$\boldsymbol{x} = P\boldsymbol{x} + Q\boldsymbol{x}\,. \tag{6}$$

The fast converging component $P\boldsymbol{x}$ can be obtained by solving $AP\boldsymbol{x} = P^tA\boldsymbol{x} = P^t\boldsymbol{b}$. The slowly component $Q\boldsymbol{x}$ is obtained from,

$$Q\boldsymbol{x} = W(W^tAW)^{-1}W^t\boldsymbol{b}. \tag{7}$$

The convergence of ICCG method depends on the condition number defined by the ratio of the largest eigenvalue λ_{max} to the smallest non-zero eigenvalue λ_{min} of system matrix. The deflation technique which replaces such the small eigenvalues with zeros in the system matrix can improve the convergence.

Here the eigenvalue of system matrix is sorted in ascending order as λ_i (i = 1,2,..,n'), n' is the number of non-zero eigenvalues. If the eigenvectors corresponding to the λ_i (i = 1,2,..,k) are chosen as $\boldsymbol{w}_i$, the condition number of the deflated system matrix becomes $\lambda_{\max}/\lambda_{k+1}$. In the typical finite element analysis, few small eigenvalues cause the large condition number. Therefore, even if a small k (k << n) is selected, the convergence of deflated method can be improved. The inverse matrix $(W^tAW)^{-1}$ in (7) can be computed by direct solvers with low computational cost because the dimension of the matrix is $k \times k$.

It takes huge computational cost to obtain the eigenvectors. As a result, the original deflated method is not practical for large scale finite element analyses. A nested geometric multigrid algorithm is introduced to solve this problem. Several meshes with different resolution are prepared. The eigenvectors on the system matrix corresponding to the coarsest mesh is computed. It can perform with low cost because of small DoF. The eigenvectors are projected onto the finer meshes by

$$\boldsymbol{w}_i^f = P\boldsymbol{w}_i^c \quad (i = 1,2,..,k) \tag{8}$$

Where $\boldsymbol{w}_i^f$ and $\boldsymbol{w}_i^c$ are the eigenvectors corresponding to the fine and coarse mesh respectively. The prolongation matrix P can be obtained by using geometric relationship between coarse and fine meshes [4]. The projected eigenvectors $\boldsymbol{w}_i^f$ is used as $\boldsymbol{w}_i$ in the present deflated method. The eigenvector corresponding to the small eigenvalue has smooth distributions. Thus the error between $\boldsymbol{w}_i^f$ and the real eigenvectors is small.

III. NUMERICAL RESULTS

We applied the present method to a shape optimization of power inductor for dc-dc converters [3]. The finite element model consists of square elements and very thin rectangular elements shown in Fig 1. Two level meshes are prepared for the multigrid. The number of divisions is $N_1 = 200,\ N_2 = 50$ for the fine mesh and $N_1 = 40,\ N_2 = 10$ for the coarse mesh. The inductance of the inductor is computed by a linear analysis for small ac signals and a non-linear analysis using Newton-Raphson method for 0.2A dc bias current. The dc and ac B-H characteristics of ferrite core are same as Ref [3]. The finite element analyses to obtain the inductance are performed on the fine mesh. The coarse mesh is used to obtain the eigenvectors. An Immune algorithm which is a stochastic optimization method is used for the optimization.

The convergence of the present method to compute the inductance is summarized in Table 1. The deflated ICCG method can improve the convergence in the non-linear as well as linear analysis. In particular, the convergence of the present method is 24% faster than that of the conventional ICCG method in the non-linear analysis. Therefore the present method can reduce the computational cost of optimization.

Other numerical examples and discussions in detail are presented in the full paper.

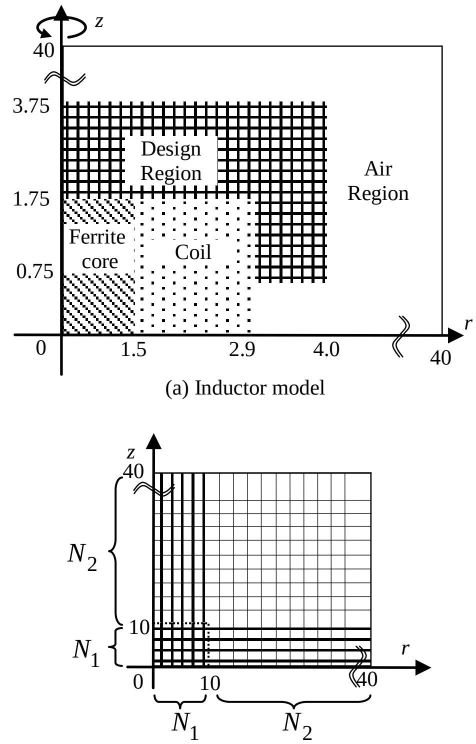

Fig. 1 Inductor model for topology optimization

TABLE I
COMPARISON OF CONVERGENCE

Method	Total number of iterations	
	Linear analysis	Non-linear analysis
Conventional ICCG	335	709
Deflated ICCG k = 4	313	540

REFERENCES

[1] H. Igarashi, K. Watanabe, "Deflation Techniques for Computational Electromagnetism: Theoretical Considerations," *IEEE Transactions on Magnetics*, Vol. 47, No. 5, pp. 1438-1441, 2011.

[2] Y. Saad, M. Yeung, J. Erhel, F. Guyomarch, "A deflated version of the conjugate gradient algorithm," *SIAM J. Sci. Comput.*, vol.21, No.5, pp. 1909-1926, 2000.

[3] K. Watanabe, F. Campelo, Y. Iijima, K. Kawano, T. Matsuo, T. Mifune and H. Igarashi, "Optimization of inductors using evolutionary algorithms and its experimental validation," IEEE *Trans. Magn.*, Vol. 46, No. 8, pp. 3393-3396, 2010.

[4] M Schinnerl, J. Schoberl, M. Kaltenbacher: "Nested multigrid methods for the fast numerical computation of 3D magnetic fields," IEEE *Trans. Magn.*, vol. 36, no 4, pp. 1557-1560, 2000.